Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 2 Mathematics

Assessment 2

June 2014

<u>TIME ALLOWED</u>: **75 minutes**

Instructions:

- ***Start each question on a new page.***
- Write your name and class at the top of this page, and on your answer booklet.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. <u>Marks may not be awarded for careless or badly arranged work.</u>
- Marks indicated within each question are a guide only and may be varied at the time of marking
- Write in blue or black pen only.
- It is suggested that you spend no more than 7 minutes on Part A.
- Approved calculators may be used.
- Standard Integrals are supplied at the rear of this paper. This is the only sheet which may be detached from any booklet.

PART A: (5 Marks)

Answers to these multiple choice should be completed on the multiple choice answer sheet supplied with your answer booklet.

All questions are worth 1 mark

1	A Primitive of $\frac{1}{\sqrt{x^2+16}}$ is A. $\frac{1}{4}\ln(x^2+16)$ B. $\frac{1}{2}\sqrt{x^2+16}$ C. $\ln(x+\sqrt{x^2+16})$ D. $\frac{1}{4}tan^{-1}(x^2+16)$
2	P(x) is a monic polynomial with real coefficients and has zeros of $2+i$, 2 and -2. P(x) = A. $x^4-4x^3+x^2+16x-20$ B. $x^4+4x^3+x^2-16x-20$ C. $x^4+2ix^3+9x^2-8ix+20$ D. $x^3-(2+i)x^2-4x+(8+4i)$
3	$\int sec^2xtanxdx \quad =$ A. $tanx + k$ B. $tan^2x + k$ C. $\frac{1}{2}tan^2x + k$ D. $\frac{1}{3}sec^3x + k$

4	If two of its roots are equal, the roots of $2x^3 + 3x^2 - 12x + 7 = 0$ are: A. $x = -1$ and $x = \frac{7}{2}$ B. $x = 1$ and $x = -\frac{7}{2}$ C. $x = -2$ and $x = -\frac{7}{8}$ D. $x = 2$ and $x = \frac{7}{8}$
5	$\int sin^3x\ dx =$ A. $\frac{1}{4}sin^4x + k$ B. $\frac{1}{4}cos^4x + k$ C. $\frac{1}{3}cos^3x - cosx + k$ D. $-cosx - \frac{1}{3}cos^3x + k$

PART B

(START EACH QUESTION ON A NEW PAGE)

QUESTION 6: (10 Marks)

Marks

1 (a) Find $\int \frac{dx}{1- \ sin^2x}$

2 (b) Evaluate $\int_2^7 \frac{x-2}{\sqrt{x+2}} \ dx$

2 (c) (i) Find values of a, b and c, so that

$$\frac{5x^2-x-2}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$$

2 (ii) Hence find $\int \frac{5x^2-x-2}{(x+1)(x^2+1)} \ dx$

3 (d) Sketch the curve, $y = ln|sinx|$ for $-2\pi \leq x \leq 2\pi$, showing all keypoints

QUESTION 7: (10 Marks) ***(Start on a new page)***

Marks

3 (a) Using t-results, or otherwise, evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+cosx}$$

3 (b) The equation $ax^4 + bx^3 + cx + d = 0$ has a triple root.

Show that $4a^2c + b^3 = 0$

1 (c) (i) Solve the equation $cos\ 5\theta + 1 = 0$ for $0 \leq \theta \leq 2\pi$

3 (ii) Using the substitution $x = cos\ \theta$,

solve $16x^5 - 20x^3 + 5x + 1 = 0$ and hence show that $cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

(__You may assume the result:__ $cos5\theta = 16cos^5\theta - 20cos^3\theta + 5cos\theta$ *)*

QUESTION 8: (10 Marks) ***(Start on a new page)***

Marks

5 (a) By using the substitution $x = 3sin\,\theta$, or otherwise, show that

$$\int \frac{x^2}{\sqrt{9-x^2}}dx = \frac{9}{2}sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + k$$

(b) The real roots of $x^3 + 4x - m = 0$ are α, β and γ.

1 (i) Find the value of $\frac{1}{\alpha^2\beta\gamma} + \frac{1}{\alpha\beta^2\gamma} + \frac{1}{\alpha\beta\gamma^2}$

1 (ii) Explain why $\frac{1}{\alpha^2\beta\gamma} = \frac{1}{m\alpha}$

3 (iii) Hence, or otherwise, find the cubic polynomial whose roots are

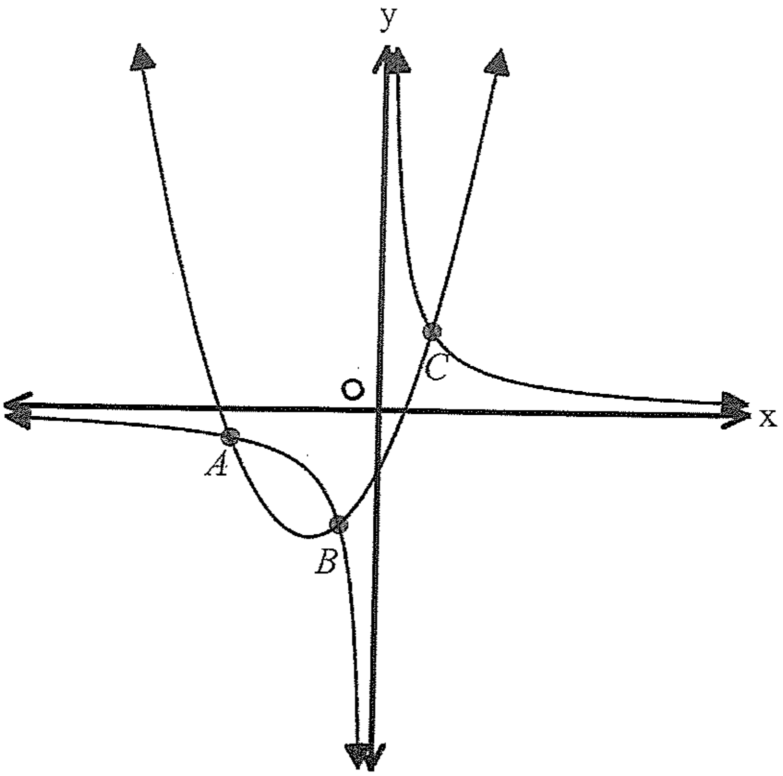
$$\frac{1}{\alpha^2\beta\gamma}, \frac{1}{\alpha\beta^2\gamma}, \text{ and } \frac{1}{\alpha\beta\gamma^2}$$

QUESTION 9: (10 Marks) *(Start on a new page)*

Marks

2 (a) Find $\int \frac{dx}{\sqrt{3+2x-x^2}}$

(b)



In the diagram above, the points A, B and C represent the points of intersection of the curve $y = x^2 + 2x - 1$ and the curve $y = \frac{1}{x}$. O is the origin

The x-values of A, B and C are α, β, and γ.

1 (i) Show that α, β, and γ satisfy $x^3 + 2x^2 - x - 1 = 0$

2 (ii) Find a polynomial with roots α^2, β^2 and γ^2

2 (iii) Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

3 (iv) Prove that $OA^2 + OB^2 + OC^2 = 11$

SOLUTIONS - YEAR 12 TERM 2
EXTENSION 2 ASSESSMENT.

Multiple Choice

1) C 2) A 3) C 4) B 5) C.

QUESTION 6:

(a) $\int \frac{dx}{1-\sin^2 x} = \int \frac{dx}{\cos^2 x}$

$= \tan x + k$

(b) $\int_2^7 \frac{x+2}{\sqrt{x+2}}\,dx - \int_2^7 \frac{4}{\sqrt{x+2}}\,dx$

$= \left[\frac{2}{3}(x+2)^{3/2}\right]_2^7 - \left[8(x+2)^{1/2}\right]_2^7$

$= 18 - \frac{16}{3} - 8(3-2)$

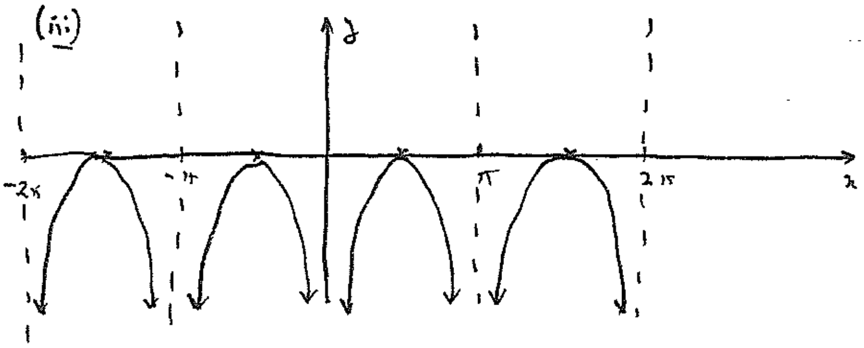
$= \frac{14}{3}$

(c) (i) $a = 2, \quad b = 3, \quad c = -4$

(ii) $\int \frac{5x^2-x-2}{(x+1)(x^2+1)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3x-4}{(x^2+1)}\,dx$

$= 2\ln(x+1) + \frac{3}{2}\ln(x^2+1) - 4\tan^{-1}x + k$

(iii)



QUESTION 7: $t = \tan\frac{x}{2}$ $\quad \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ $\quad x = 0, \ t = 0$

$\therefore dx = \frac{2}{\sec^2\frac{x}{2}}\,dt$ $\quad x = \frac{\pi}{2}, \ t = 1$

$= \frac{2}{1+t^2}\,dt$

$\therefore \int_0^1 \frac{\frac{2}{1+t^2}\,dt}{2 + \frac{1-t^2}{1+t^2}}$

$= \int_0^1 \frac{2\,dt}{2+2t^2+1-t^2}$

$= 2\int_0^1 \frac{dt}{t^2+3}$

$= 2\cdot\frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= \frac{\pi}{3\sqrt{3}}$

(b) $P'(x) = 4ax^3 + 3bx^2 + c$

$P''(x) = 12ax^2 + 6bx$.

For triple roots, $P''(x) = 0$

$\therefore 6x(2ax + b) = 0$

$\therefore x = 0$ or, $x = -b/2a$

But $P(x) \neq 0$ $\therefore$ triple root is $-b/2a$

$\therefore P'(-b/2a) = 4a(-b/2a)^3 + 3b(-b/2a)^2 + c = 0$

$\therefore -\frac{b^3}{2a^2} + \frac{3b^3}{4a^2} + c = 0$

$b^3 + 8a^2c = 0$

(c)(i) Since $0 \le \theta \le 2\pi$, $0 \le 5\theta \le 10\pi$,

$\therefore 5\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi$.

$\therefore \theta = \pi/5, 3\pi/5, \pi, 7\pi/5, 9\pi/5$

(ii) Let $x = \cos\theta$

$\therefore 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 1 = 0$

ie. $\cos 5\theta + 1 = 0$

$\therefore \theta = \pi/5, 3\pi/5, \pi, 7\pi/5, 9\pi/5$ (from above)

$\therefore x = \cos\theta = \cos\pi/5, \cos 3\pi/5, \cos\pi, \cos 7\pi/5, \cos 9\pi/5$

and $\cos 7\pi/5 = \cos 3\pi/5$, $\cos 9\pi/5 = \cos\pi/5$.

$\therefore$ By sum of roots,

$\cos\pi/5 + \cos 3\pi/5 - 1 + \cos 3\pi/5 + \cos\pi/5 = 0$

$\therefore 2(\cos\pi/5 + \cos 3\pi/5) = 1$

$\therefore \cos\pi/5 + \cos 3\pi/5 = +1/2$

QUESTION 8:

(a) $x = 3\sin\theta \Rightarrow dx/d\theta = 3\cos\theta$

$dx = 3\cos\theta\, d\theta$

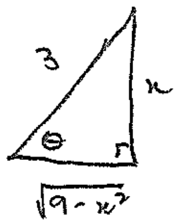
$$\therefore \int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta\, d\theta$$

$$= \int 9\sin^2\theta\, d\theta$$

$$= \frac{9}{2}\int 1 - \cos 2\theta\, d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + k$$

$$= 9/2\theta - \frac{9}{2}\sin\theta\cos\theta + k$$

Now 

(from $x = 3\sin\theta$)

gives $\cos\theta = \sqrt{9-x^2}/3$

$\therefore$ Integral $= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{9}{2}\cdot\frac{x}{3}\cdot\sqrt{9-x^2}/3 + k$

$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + k$

(b) (i) $\begin{cases}\alpha+\beta+\gamma = 0 \\ \alpha\beta+\alpha\gamma+\beta\gamma = 4 \\ \alpha\beta\gamma = m\end{cases}$

$\therefore \frac{1}{\alpha^2\beta\gamma} + \frac{1}{\alpha\beta^2\gamma} + \frac{1}{\alpha\beta\gamma^2} = \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha^2\beta^2\gamma^2}$

$= 4/m^2$

(ii) Since $\alpha\beta\gamma = m$

then $\alpha^2\beta\gamma = \alpha m$

$\Rightarrow \frac{1}{\alpha^2\beta\gamma} = \frac{1}{\alpha m}$

(iii) Polynomial is $x^3 - (\text{sum of roots})x^2 + (\text{sum in pairs})x - \text{Product} =$

Sum in pairs $= \frac{1}{\alpha^3\beta^2\gamma^2} + \frac{1}{\alpha^2\beta^3\gamma^2} + \frac{1}{\alpha^2\beta^2\gamma^3}$

$= \frac{\alpha+\beta+\gamma}{\alpha^3\beta^3\gamma^3} = 0$ (from above)

Product $= \frac{1}{\alpha^4\beta^4\gamma^4}$

$= 1/m^4$

from part (i)

$\therefore$ Polynomial is $x^3 - \frac{4}{m^2}x^2 + 0x - 1/m^4 = 0$

$\therefore m^4x^3 - 4m^2x^2 - 1 = 0$.

QUESTION 9:

(a) $\int\frac{dx}{\sqrt{3+2x-x^2}} = \int\frac{dx}{\sqrt{3-(x-1)^2+1}}$

$= \int\frac{dx}{\sqrt{4-(x-1)^2}}$

$= \sin^{-1}\left(\frac{x-1}{2}\right) + k$

(b) (i) Intersection of $y = 1/x$ and $y = x^3 - 2x - 1$

ie $x^3 - 2x - 1 = 1/x$

$\therefore x^4 - 2x^2 - x = 1$

$\therefore x^4 - 2x^2 - x - 1 = 0$

(ii) $P(\sqrt{x}) = (\sqrt{x})^3 + 2(\sqrt{x})^2 - (\sqrt{x}) - 1 = 0$

$\sqrt{x}(x-1) = 1 - 2x$

$\therefore x(x-1)^2 = (1-2x)^2$

$\therefore x^3 - 2x^2 + x = 1 - 4x + 4x^2$

$\therefore x^3 - 6x^2 + 5x - 1 = 0$

(iii) In above, $\alpha^2 + \beta^2 + \gamma^2 = 6$

$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = 5$

$\alpha^2\beta^2\gamma^2 = 1$

$$\therefore \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{\alpha^2\beta^2\gamma^2}$$

$= 5.$

(iv) If m is the foot of the perpendicular from A to the x-axis.

$OA^2 = \alpha^2 + (1/\alpha)^2$ by distance formula.

$$\therefore OA^2 + OB^2 + OC^2 = \alpha^2 + (1/\alpha)^2 + \beta^2 + \left(\frac{1}{\beta}\right)^2 + \gamma^2 + \left(\frac{1}{\gamma}\right)^2$$

$$= \alpha^2 + \beta^2 + \gamma^2 + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)$$

$= 6 + 5$

$= 11$